Solution 8

1. Define the operator norm of an $n \times n$-matrix A by

$$\|A\| = \sup\{|Ax| : \ |x| \le 1\} \ ,$$

where $|x|$ is the Euclidean norm of $x \in \mathbb{R}^n$.

(a) Show that

$$\|A\| = \sup\left\{\frac{|Ax|}{|x|} \ : \ x \neq 0\right\} \ .$$

(b) Show that

$$\|A\| = \inf\{M : \ |Ax| \le M|x|, \ \forall x\} \ .$$

(c) Show that $\|A\|^2$ is equal to the largest eigenvalue of the symmetric matrix A^tA (A^t is the transpose of A).

Solution. (a) Use $|Ax|/|x| = |Ay|$ for $y = x/|x|$ which is a unit vector.

(b) Let M satisfy $|Ax| \le M|x|$ for all x. Restrict it to $|x| \le 1$ and take sup to get $\|A\| = \sup |Ax| \le M$. On the other hand, $\|A\|$ satisfies $|Ax| \le \|A\||x|$ for all x, hence $\|A\|$ is one of these M. It shows that inf can be replaced by min.

(c) $|Ax|^2 =< Ax, Ax >=< A^tAx, x >$ where $< \cdot, \cdot >$ is the Euclidean product. Recall that A^tA is a non-negative matrix and $\max\{< A^tAx, x >: \ |x| \le 1\}$ gives the largest eigenvalue of this non-negative matrix.

2. There are other norms defined on $\mathbb{R}^n$ other than the Euclidean one. For example, now consider $\|x\|_1 = \sum_{k=1}^n |x_k|$. For an $n \times n$-matrix A, define

$$\|A\|_1 = \sup\{\|Ax\|_1 : \|x\|_1 \le 1\} \ .$$

(a) Show that

$$\|A\|_1 = \inf\{M : \ \|Ax\|_1 \le M\|x\|_1, \ \forall x\} \ .$$

(b) Show that

$$\|A\|_1 = \max_j \sum_{i=1}^n |a_{ij}| \ .$$

(c) Show that the conclusion in Problem 8, Ex 7, still holds when the condition $\sum_{i,j} a_{ij}^2 < 1$ is replaced by the weaker condition $\|A\|_1 < 1$.

Solution. (a) For $x \neq 0$, $x/\|x\|_1$ has norm equal to 1. So $\|A(x/\|x\|_1)\|_1 \le \|A\|_1$ which implies $|Ax|_1 \le \|A\|_1\|x\|_1$ by linearity. Next, if $\|Ax\|_1 \le M\|x\|_1$, take $x, \|x\|_1 \le 1$ in this inequality and then take sup over all such x. By the def of $\|A\|_1$ we get $\|A\|_1 \le M$. So (a) holds.

(b) Let $\alpha = \max_j \sum_{i=1}^n |a_{ij}|$. Taking $x = e_j$ in $\|Ax\|_1 \le M\|x\|_1$, we get $|\sum_i a_{ij}| \le M$ for all j. On the other hand, it is clear that $\|Ax\|_1 \le \alpha\|x\|_1$, so by (a) $\alpha = \|A\|_1$.

(c) Yes, just change the Euclidean norm to $\| \cdot \|_1$ and follow the same proof.

This problem shows, when different norms are used in the Euclidean space, the notion of smallness changes accordingly. The norm $\|A\|_1$ applies to some situation when $\|A\| \ge 1$ but $\|A\|_1 < 1$.

3. Let f be continuously differentiable on $[a, b]$. Show that it has a differentiable inverse if and only if its derivative is either positive or negative everywhere.

   Solution. $\Rightarrow$. Let g be the inverse of f. When g is differentiable, we can use the chain rule in the relation $g(f(x)) = x$ to get $g'(f(x))f'(x) = 1$, which implies that $f'(x)$ never vanishes. Since f' is continuous, if $f'(x_0) > 0$ at some x_0, we claim f' is positive everywhere. Suppose $f'(x_1) < 0$ at some x_1, by continuity $f'(x_2) = 0$ at some x_2 between x_0 and x_1, contradiction holds. Hence f' is positive everywhere. Similarly, it is negative everywhere when it is negative at some point.

   $\Leftarrow$. Let us assume f' is always positive (the other case can be treated similarly.) Let $x < y$ in $[a, b]$. By the mean value theorem, there is some $z \in (x, y)$ such that $f(y) - f(x) = f'(z)(y - x) > 0$, so f is strictly increasing. According to an old result in 2050, a continuous, strictly increasing function maps $[a, b]$ to the interval $[f(a), f(b)]$ and its inverse g is continuous. Then we can use the Carathedory Criterion in 2060 to show that g is differentiable and, in fact, satisfies $g'(f(x)) = 1/f'(x)$.

4. Consider the function
$$f(x) = \frac{1}{2}x + x^2 \sin\frac{1}{x}, \quad x \neq 0,$$
   and set $f(0) = 0$. Show that f is differentiable at 0 with $f'(0) = 1/2$ but it has no local inverse at 0. Does it contradict the inverse function theorem?

   Solution. $|f(x) - f(0) - (1/2)x| = |x^2 \sin(1/x)| = O(x^2)$, hence f is differentiable at 0 with $f'(0) = 1/2$. Let $x_k = 1/2k\pi, y_k = 1/(2k\pi + 1)$, then $f'(x_k) = -1/2, f'(y_k) = 3/2$. Then it is clear that f is not injective in $I_k = (y_k, x_k)$. Since any neighborhood of 0 must include contain some I_k, this shows that f it has no local inverse at 0. It does not contradict the inverse function theorem because f' is not continuous at 0.

   Note. This problem shows that the C^1-condition is needed in the Inverse Function Theorem.

5. Consider the mapping from $\mathbb{R}^2$ to itself given by $f(x, y) = x - x^2,\ g(x, y) = y + xy$. Show that it has a local inverse at $(0, 0)$. And then write down the inverse map so that its domain can be described explicitly.

   Solution. Let $u = x - x^2, v = y + xy$. The Jacobian determinant is 1 at $(0, 0)$ so there is an inverse in some open set containing $(0, 0)$. Now we can describe it explicitly as follows. From the first equation we have
$$x = \frac{1 \pm \sqrt{1 - 4u}}{2}.$$
   From $u(0, 0) = 0$ we must have
$$x = \frac{1 - \sqrt{1 - 4u}}{2}.$$
   Then
$$y = \frac{v}{1 + x} = \frac{2v}{1 - \sqrt{1 - 4u}}.$$
   We see that the largest domain in which the inverse exists is $\{(u, v) : u \in (0, 1/4), v \in \mathbb{R}\}$.

6. Let F be a continuously differentiable map from the open $U \subset \mathbb{R}^n$ to $\mathbb{R}^n$ whose Jacobain determinant is non-vanishing everywhere. Prove that it maps every open set in U to an open set, that is, F is an open map. Does its inverse $F^{-1} : F(U) \to U$ always exist?

Solution. Let $G \subset U$ be open and $y_0 = F(x_0), x_0 \in G$. By the inverse function theorem, there are balls $B_\delta(x_0) \subset G$ and $B_R(y_0)$ such that $B_R(y_0) \subset F(B_\delta(x_0))$, so $F(G)$ is open. That is, F is an open map. On the other hand, the change from the rectangular coordinates to the polar coordinators is an example where an everywhere locally invertible map is not globally invertible.